



The Open
University

MST125

Essential mathematics 2

Exercise Booklet 12

1 Principles of counting

Exercise 1

Let $S = \{1, 2, \dots, n\}$, where n is an integer greater than 6.

- How many subsets of S contain the element 1, but do not contain the elements 3 and 4?
- How many subsets of S contain at least one of the elements 1 and 2, and at least two of the elements 3, 4 and 5?
- How many subsets of S are subsets of one of $\{1, 2, 3, 4\}$ and $\{5, 6, \dots, n\}$?

Exercise 2

- How many positive integers less than 900 end in the digit 4 in their decimal representations?
- How many positive integers less than 900 contain at least one 5 in their decimal representations?
- How many positive integers less than 900 contain at least one of the digits 2 and 3 in their decimal representations?
- How many 3-digit positive integers less than 900 are there whose decimal representation contains a digit that appears more than once?
- How many positive integers less than 900 contain both of the digits 2 and 3 in their decimal representations?

Hint: Let $U = \{1, 2, \dots, 899\}$. Let A be the subset of U containing those integers whose decimal representations contain at least one 2, and similarly let B be the subset of U containing those integers whose decimal representations contain at least one 3. The answer that we are seeking is $|A \cap B|$. The answer to part (c) is $|A \cup B|$, and the answer to part (b) essentially gives $|A|$ and $|B|$. Can you connect, using a Venn diagram as in Unit 3 of MST124, the sizes of the sets A , B and $A \cup B$ with the size of $A \cap B$?

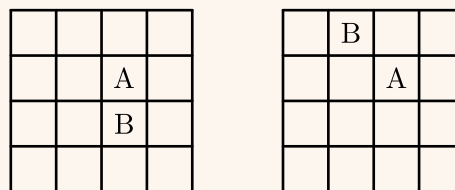
Exercise 3

The set A has 2 elements, and the set B has 5 elements. Find the number of each of the following.

- Functions from A to B .
- One-to-one functions from A to B .
- Functions from A to B whose image set is all of B . (Such a function is said to be *onto* B .)
- Functions from B to A .
- One-to-one functions from B to A .
- Functions from B to A whose image set is all of A .

Exercise 4

- How many different ways are there to place an A and a B in adjacent squares of a 4×4 grid, where two squares are taken to be adjacent if they meet either at an edge or at a corner? Two examples are shown below.



- How many different ways are there to place two As in adjacent squares of a 4×4 grid?

2 Sequences, permutations and combinations

Exercise 5

A chess club has 12 members.

- (a) Every month, each member plays exactly one match with each other member. How many matches are there in a month?
 - (b) A ranking list is produced at the end of each month based on that month's matches. How many possibilities are there for the top three players in ranked order? Assume that the ranking doesn't allow for members tying for a place.
 - (c) The top three players at the end of April are sent as a team to play a team from a neighbouring club. How many ways are there of forming such a team? Assume that the order of the three team members is not important.
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Exercise 6

A game involves placing four coloured pegs in a row on a pegboard. Many pegs of six different colours are supplied.

- (a) How many possibilities are there for the row of pegs?
 - (b) How many possibilities are there for a row of pegs with no repeated colours?
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Exercise 7

Consider the sequences of four letters that can be made using the first ten letters of the alphabet, A, B, ..., J. Examples include JBCG and BABB.

- (a) How many such sequences are there altogether?
- (b) How many of the sequences start with one of the first five letters of the alphabet?
- (c) How many of the sequences start and end with the same letter?

- (d) How many of the sequences contain the letter A?
 - (e) How many of the sequences contain at least one of the letters A and B?
 - (f) How many of the sequences contain a letter repeated at least once?
 - (g) How many of the sequences contain at least two different letters?
 - (h) How many of the sequences contain exactly two different letters?
 - (i) How many of the sequences contain at most two different letters?
 - (j) How many of the sequences contain at least three occurrences of the same letter?
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Exercise 8

A choir consists of five sopranos, six altos, three tenors and four basses.

- (a) How many ways are there to choose a single singer from each of the voice groups to form a quartet (that is, a set of four singers)?
 - (b) How many ways are there to choose six singers to form a sextet (that is, a set of six singers)?
 - (c) How many ways are there to choose a pair of singers from each of the voice groups to form an octet (that is, a set of eight singers)?
 - (d) How many ways are there to choose one singer from two of the voice groups to sing a duet?
 - (e) An octet is to be arranged on a stage in a line going from left to right, with two sopranos on the left, then two altos to their right, then two tenors to their right, and finally two basses at the far right. In how many ways can the singers be chosen and arranged in this pattern?
 - (f) The entire choir is to be arranged on a stage in a line going from left to right, with each voice group standing together. So, for instance, one might have all the sopranos in a line on the left, then a line of all the tenors to their right, all the altos to their right and all the basses at the far right. How many ways are there to arrange all the singers on the stage?
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Exercise 9

- (a) In how many ways can the 5-element set $S = \{1, 2, 3, 4, 5\}$ be written as the union of a 2-element subset and a disjoint 3-element subset?
- (b) Consider the 4-element set $S = \{1, 2, 3, 4\}$. What, if anything, is wrong with the following method for counting how many ways S can be written as the union of two disjoint 2-element subsets?

We can choose the first 2-element subset A of S in ${}^4C_2 = 6$ ways. As S has four elements, the complement $\overline{A}$ of A also has two elements and is disjoint from A , and $S = A \cup \overline{A}$. Thus the set $S = \{1, 2, 3, 4\}$ can be written as the union of two disjoint 2-element subsets in the same number of ways as the number of 2-element subsets of S , namely six.

- (c) In how many ways can the 6-element set $S = \{1, 2, 3, 4, 5, 6\}$ be written as the union of three pairwise disjoint 2-element subsets?
-

Exercise 10

Find the number of functions from a 4-element set A to a 3-element set B whose image set is all of B .

Exercise 11

A school's seven language teachers comprise three French specialists, two German specialists and two Spanish specialists. How many ways are there to choose a team of four of the teachers for a school event, if the team must include at least one specialist in each language?

Exercise 12

On each of the five days of the school week, a school runs four special-interest clubs immediately after school for its pupils. All twenty clubs are different. Pupils are invited to choose up to three clubs from the twenty available, but cannot choose two that take place on the same day of the week. How many different ways are there for a pupil to choose exactly three clubs?

Exercise 13

Three standard six-sided dice are rolled. Calculate the probability of each of the following events.

- (a) There are exactly three 5s.
(b) There is at least one 5.
(c) The total score is 5.
(d) The total score is at least 5.
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Exercise 14

A coin is tossed six times. Calculate the probability of each of the following events.

- (a) The coin lands showing tails every time.
(b) There is at least one tail.
(c) There is exactly one tail.
(d) There are at least two tails.
-

3 First-order recurrence systems

Exercise 15

Find a closed form for each of the following infinite first-order recurrence sequences.

- (a) The sequence (x_n) with recurrence system

$$x_1 = -4, \quad x_n = \frac{5}{2}x_{n-1} + 8 \quad (n = 2, 3, 4, \dots).$$

- (b) The sequence (u_n) with recurrence system

$$u_1 = 6, \quad u_n = -\frac{1}{2}u_{n-1} + 3 \quad (n = 2, 3, 4, \dots).$$

Exercise 16

A suspicious-sounding investment scheme is organised as follows. For an initial investment of £1.25 at the start of day 1, each subsequent morning the scheme will double the investment and then make a daily deduction of £1 as a service charge.

- (a) Let x_n be the investment in pounds at the start of day n , for $n \geq 1$. Find a recurrence system for x_n .
- (b) Find a closed form for this recurrence system.
- (c) Will the investment ever exceed £1000? If so, on which day will this first happen?
- (d) What would happen if the initial investment were changed to £1?

4 Second-order recurrence systems

Exercise 17

Find closed forms for the sequences given by the following recurrence systems.

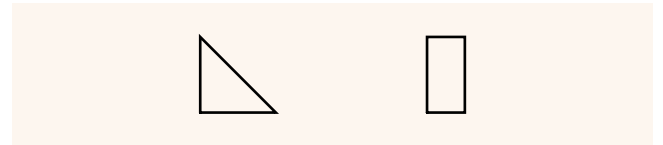
- (a) $u_0 = 9, \quad u_1 = -10,$
 $u_n = -\frac{11}{3}u_{n-1} + \frac{4}{3}u_{n-2} \quad (n = 2, 3, 4, \dots)$

- (b) $u_0 = 5, \quad u_1 = -1,$
 $u_n = -\frac{2}{3}u_{n-1} - \frac{1}{9}u_{n-2} \quad (n = 2, 3, 4, \dots)$

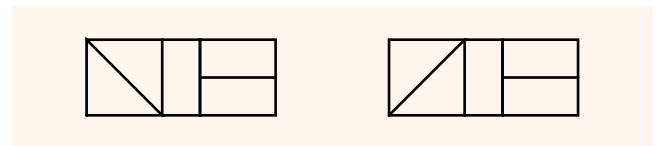
- (c) $x_0 = -1, \quad x_1 = 14,$
 $x_n = \frac{7}{2}x_{n-1} + 2x_{n-2} \quad (n = 2, 3, 4, \dots)$

Exercise 18

A rectangular path is to be laid 2 feet wide and n feet long, using stone slabs. Two sorts of paving slab are available: a right-angled triangular slab with both shorter sides of length 2 ft, and a rectangular slab measuring 1 ft $\times$ 2 ft, as shown below.



For example, here are two possible paths that are 2 ft wide and 5 ft long.



Let u_n denote the number of ways to lay the 2 ft $\times$ n ft path.

- (a) Calculate u_1 and u_2 .
- (b) Find a recurrence system for the number u_n of ways to lay the path for $n \geq 3$.
- (c) Using your recurrence system, find a closed form for the number of ways to lay the path.

Hint: use the values u_1 and u_2 , and the recurrence relation, to obtain a value for u_0 , to make the calculation of the constants A and B a bit easier.

Solutions to exercises

Solution to Exercise 1

- (a) Every subset of S of the required form consists of the element 1 together with a subset of $\{2, 5, 6, \dots, n\}$.

So we need to count the number of subsets of $\{2, 5, 6, \dots, n\}$. Since this set contains $n - 3$ elements, it has 2^{n-3} subsets.

Therefore there are 2^{n-3} subsets of S containing the element 1 but not the elements 3 and 4.

- (b) Every subset of S of the required form consists of a non-empty subset of $\{1, 2\}$, together with a subset of $\{3, 4, 5\}$ containing at least two elements, together with a subset of $\{6, 7, \dots, n\}$. So each subset of the required form can be found by making three choices.

First, choose a non-empty subset of $\{1, 2\}$. There are three options for this choice, namely all subsets of $\{1, 2\}$ except $\emptyset$.

Second, choose a subset of $\{3, 4, 5\}$ containing at least two elements. There are four options for this choice, namely $\{3, 4\}$, $\{3, 5\}$, $\{4, 5\}$ and $\{3, 4, 5\}$.

Finally, choose a subset of $\{6, 7, \dots, n\}$. As this set contains $n - 5$ elements, there are 2^{n-5} such subsets, so there are 2^{n-5} options for this choice.

Since the number of options for each choice is fixed, we can use the multiplication principle. It follows that the number of subsets of S that contain at least one of the elements 1 and 2 and at least two of the elements 3, 4 and 5 is

$$3 \times 4 \times 2^{n-5} = 3 \times 2^{n-3}.$$

- (c) The 4-element set $\{1, 2, 3, 4\}$ has 2^4 subsets. The set $\{5, 6, \dots, n\}$ has $n - 4$ elements, so has 2^{n-4} subsets. However, the empty set $\emptyset$ is counted as a subset in both lists, so there are

$$2^4 + 2^{n-4} - 1 = 15 + 2^{n-4}$$

distinct subsets of one of $\{1, 2, 3, 4\}$ and $\{5, 6, \dots, n\}$.

(A fuller argument, exploiting the addition principle, is as follows. Let X be the required set of subsets of one of $\{1, 2, 3, 4\}$ and $\{5, 6, \dots, n\}$. Let X_1 be $\{\emptyset\}$. Let X_2 be the set of all non-empty subsets of $\{1, 2, 3, 4\}$, and let X_3 be the set of all non-empty subsets of $\{5, 6, \dots, n\}$. Then $X = X_1 \cup X_2 \cup X_3$, and X_1 , X_2 and X_3 are pairwise disjoint, so the addition principle can be used. We have

$$\begin{aligned} |X_1| &= 1, \\ |X_2| &= 2^4 - 1 = 15, \\ |X_3| &= 2^{n-4} - 1. \end{aligned}$$

So, by the addition principle,

$$\begin{aligned} |X| &= |X_1| + |X_2| + |X_3| \\ &= 1 + 15 + 2^{n-4} - 1 \\ &= 15 + 2^{n-4}. \end{aligned}$$

Thus there are $15 + 2^{n-4}$ subsets of S that are subsets of one of $\{1, 2, 3, 4\}$ and $\{5, 6, \dots, n\}$.)

Solution to Exercise 2

- (a) Think of every integer of the required form as a 3-digit number, by including leading zeros if necessary. Then there are nine options (namely $0, 1, \dots, 8$) for the first digit (the hundreds digit), ten options (namely $0, 1, \dots, 9$) for the second digit (the tens digit), and one option (namely 4) for the third digit (the units digit). Therefore, by the multiplication principle, the number of integers of the required form is $9 \times 10 \times 1 = 90$.

(Alternatively, the numbers of the required form are those that can be written as $10n + 4$ for some integer n in the range $0 \leq n \leq 89$. So there are 90 such numbers.)

- (b) We need to count the elements of

$S =$ set of all positive integers less than 900 containing at least one 5.

Let $U = \{1, 2, \dots, 899\}$. Then $|U| = 899$ and S is a subset of U .

The complement of S in U is

$\overline{S} =$ set of all positive integers less than 900 not containing a 5.

To count the set $\overline{S}$, think of every integer in $\overline{S}$ as a 3-digit number, by including leading zeros if necessary. Then there are eight options (any digit except 5 and 9) for the first digit, and nine options (any digit except 5) for each of the other two digits, except that this count includes the integer $0 = 000$, which is not in $\overline{S}$. Hence, by the multiplication principle and the subtraction principle,

$$|\overline{S}| = 8 \times 9 \times 9 - 1 = 647.$$

Therefore, by the subtraction principle again,

$$|S| = |U| - |\overline{S}| = 899 - 647 = 252.$$

That is, there are 252 numbers of the required form.

- (c) We need to count the elements of

S = set of all positive integers
less than 900
containing at least one 2 or 3.

Let $U = \{1, 2, \dots, 899\}$. Then $|U| = 899$ and S is a subset of U .

The complement of S in U is

$\overline{S}$ = set of all positive integers
less than 900 containing no 2 or 3.

To count the set $\overline{S}$, think of every integer in $\overline{S}$ as a 3-digit number, by including leading zeros if necessary. Then there are seven options (any digit except 2, 3 and 9) for the first digit, and eight options (any digit except 2 and 3) for each of the other two digits, except that this count includes the integer $0 = 000$, which is not in $\overline{S}$. Hence, by the multiplication principle and the subtraction principle,

$$|\overline{S}| = 7 \times 8 \times 8 - 1 = 447.$$

Therefore, by the subtraction principle again,

$$|S| = |U| - |\overline{S}| = 899 - 447 = 452.$$

That is, there are 452 numbers of the required form.

- (d) We need to count the elements of

S = set of 3-digit positive integers
containing a repeated digit.

Let $U = \{100, 101, \dots, 899\}$. Then $|U| = 800$, and S is a subset of U .

The complement of S in U is

$\overline{S}$ = set of 3-digit positive integers
not containing a repeated digit.

Consider any integer in $\overline{S}$. There are eight options (any digit except 0 and 9) for the first digit, nine options (any digit except the digit chosen for the first digit) for the second digit, and eight options (any digit except the digits chosen for the first and second digits) for the third digit. So, by the multiplication principle,

$$|\overline{S}| = 8 \times 9 \times 8 = 576.$$

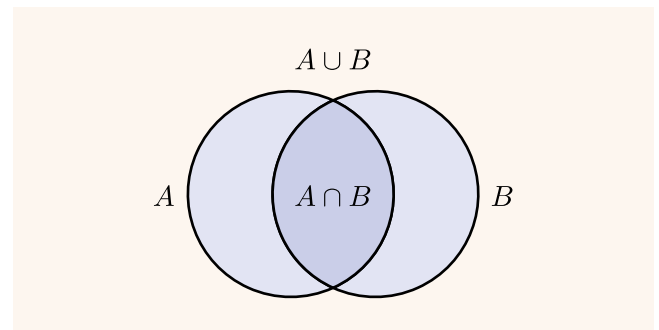
Therefore, by the subtraction principle,

$$|S| = |U| - |\overline{S}| = 800 - 576 = 224.$$

That is, there are 224 numbers of the required form.

- (e) From the Venn diagram below, you can see that the sum $|A| + |B|$ counts the elements of $A \cap B$ twice. So

$$|A| + |B| = |A \cup B| + |A \cap B|.$$



From part (c), $|A \cup B| = 452$. If part (b) had asked how many positive integers less than 900 contain at least one 2, rather than at least one 5, in their decimal representations, then its answer would be unchanged; likewise if a 3 had been mentioned instead of a 5. Thus, from the answer to part (b), we have $|A| = |B| = 252$. So

$$\begin{aligned} |A \cap B| &= |A| + |B| - |A \cup B| \\ &= 252 + 252 - 452 \\ &= 52. \end{aligned}$$

That is, there are 52 positive integers less than 900 that contain both of the digits 2 and 3 in their decimal representations.

(The method used here, based on the Venn diagram above, is an instance of the *inclusion-exclusion principle*.)

Solution to Exercise 3

- (a) For each element a of A , there is a choice of five elements of B for its image $f(a)$ under a function f . So, as A has two elements, by the multiplication principle there are

$$5 \times 5 = 5^2 = 25$$

functions from A to B .

- (b) Label the two elements of A as a_1 and a_2 . For a one-to-one function f , there is a choice of five elements of B for $f(a_1)$, and then a choice of four elements of B for $f(a_2)$. So, by the multiplication principle, there are $5 \times 4 = 20$ one-to-one functions from A to B .
- (c) There are not enough elements in A for the image set of a function f from A to B to be all of B , so there are no functions from A to B with image set B .
- (d) For each element b of B there is a choice of two elements of A for its image $f(b)$ under a function f . So, as B has five elements, by the multiplication principle there are

$$2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32$$

functions from B to A .

- (e) There are not enough elements in A to provide distinct images for a one-to-one function from the 5-element set B , so there are no one-to-one functions from B to A .
- (f) It is easy to count the number of functions from B to A whose image set is *not* all of A . Each such function maps all five elements of B to the same element a of A . As A has two elements, there are two options for this element a , so there are two such functions. Hence, by the subtraction principle, as there are 32 functions from B to A , there are

$$32 - 2 = 30$$

functions from B to A with image set A .

Solution to Exercise 4

- (a) If the A is placed in a corner square, then there are four ways to place the A, and then three ways to place the B. So, by the multiplication principle, there are $4 \times 3 = 12$ ways to place the two letters in this case.

If the A is placed in a square that lies on an edge but not in a corner, then there are eight ways to place the A, and then five ways to place the B. So, by the multiplication principle, there are $8 \times 5 = 40$ ways to place the two letters in this case.

If the A is placed in an interior square, then there are four ways to place the A, and then eight ways to place the B. So, by the multiplication principle, there are $4 \times 8 = 32$ ways to place the two letters in this case.

Therefore, by the addition principle, the total number of ways to place the two letters is

$$12 + 40 + 32 = 84.$$

- (b) Each of the required ways of placing the two As can be obtained by taking one of the ways of placing an A and a B considered in part (a), and replacing the B by a A. However, each way of placing the two As is obtained in this way from two different ways of placing the A and the B. So the answer to this part of the question is half of the answer obtained to part (a). That is, the total number of ways to place the two As is $\frac{1}{2} \times 84 = 42$.

Solution to Exercise 5

- (a) Each pair of members has a match each month, so the number of matches is just the number of combinations of 12 members taken two at a time, which is

$${}^{12}C_2 = \frac{12 \times 11}{2 \times 1} = 66.$$

- (b) The number of possibilities is the number of permutations of 12 members taken three at a time, which is

$${}^{12}P_3 = 12 \times 11 \times 10 = 1320.$$

- (c) The number of possible teams is the number of combinations of 12 members taken three at a time, which is

$${}^{12}C_3 = \frac{12 \times 11 \times 10}{3 \times 2 \times 1} = 220.$$

Solution to Exercise 6

- (a) The number of possibilities is $6^4 = 1296$.
- (b) The number of possibilities is

$${}^6P_4 = 6 \times 5 \times 4 \times 3 = 360.$$

Solution to Exercise 7

- (a) The total number of such sequences is

$$10^4 = 10\,000.$$

- (b) For a sequence that starts with one of the first five letters of the alphabet, there are five options for the first letter, and ten options for each of the other three letters. Hence, by the multiplication principle, the total number of such sequences is

$$5 \times 10 \times 10 \times 10 = 5000.$$

- (c) For a sequence that starts and ends with the same letter, there are ten options for each of the first three letters, and one option for the final letter (namely the same letter as the first letter). Hence, by the multiplication principle, the total number of such sequences is

$$10 \times 10 \times 10 \times 1 = 1000.$$

- (d) To answer this part, we start by counting the number of sequences that do not contain the letter A. The total number of such sequences is 9^4 , since there are nine options for each of the four letters. By the subtraction principle, since there are 10^4 sequences altogether, the number of sequences that contain the letter A is

$$10^4 - 9^4 = 10\,000 - 6561 = 3439.$$

- (e) To answer this part, we start by counting the number of sequences that contain neither of the letters A and B. The total number of such sequences is 8^4 , since there are eight options for each of the four letters. By the subtraction principle, since there are 10^4 sequences altogether, the number of sequences that contain at least one of the letters A and B is

$$10^4 - 8^4 = 10\,000 - 4096 = 5904.$$

- (f) To answer this part, we start by counting the number of sequences that do not contain any repeated letters. The number of such sequences is $^{10}P_4$. By the subtraction principle, since there are 10^4 sequences altogether, the number of sequences that contain a repeated letter is

$$\begin{aligned} 10^4 - ^{10}P_4 &= 10\,000 - 10 \times 9 \times 8 \times 7 \\ &= 10\,000 - 5040 \\ &= 4960. \end{aligned}$$

- (g) To answer this part, we start by counting the number of sequences that contain only one letter. Since there are ten options for the letter, the number of such sequences is ten. By the subtraction principle, since there are 10^4 sequences altogether, the number of sequences that contain at least two different letters is

$$10^4 - 10 = 10\,000 - 10 = 9990.$$

- (h) Consider choosing a sequence that contains exactly two different letters. We can start by choosing the two different letters: the number of options is $^{10}C_2$. Then we can choose a sequence that contains these two letters and no others. The number of sequences that contain no letters other than the two chosen letters is 2^4 . However, two of these sequences contain only one letter. So the number of sequences that contain both of the chosen letters and no other letters is $2^4 - 2$. Hence, by the multiplication principle, the number of sequences that contain exactly two different letters is

$$\begin{aligned} ^{10}C_2 \times (2^4 - 2) &= \frac{10 \times 9}{2 \times 1} \times (16 - 2) \\ &= 45 \times 14 \\ &= 630. \end{aligned}$$

- (i) By part (h), the number of sequences that contain exactly two different letters is 630. Also, the number of sequences that contain exactly one letter is ten, since there are ten options for the letter. So, by the addition principle, the number of sequences that contain at most two different letters is

$$630 + 10 = 640.$$

- (j) We can count separately the number of sequences that contain exactly four occurrences of the same letter, and the number of sequences that contain exactly three occurrences of the same letter, and then apply the addition principle.

The number of sequences that contain exactly four occurrences of the same letter is ten.

Exercise Booklet 12

Now consider choosing a sequence that contains exactly three occurrences of the same letter. There are ten options for the letter that occurs three times, and then nine options for the letter that occurs once. Once the choices of the two letters have been made, there are four options for the sequence (for example, if the letter that occurs three times is A and the letter that occurs once is B, then the four possible sequences are BAAA, ABAA, AABA and AAAB). Hence, by the multiplication principle, the number of sequences that contain exactly three occurrences of the same letter is

$$10 \times 9 \times 4 = 360.$$

Therefore, by the addition principle, the number of sequences that contain at least three occurrences of the same letter is

$$10 + 360 = 370.$$

Solution to Exercise 8

- (a) There are five options for the soprano, six options for the alto, three options for the tenor, and four options for the bass. So, by the multiplication principle, there are

$$5 \times 6 \times 3 \times 4 = 360$$

ways of choosing a single singer from each of the voice groups to form a quartet.

- (b) There are

$$5 + 6 + 3 + 4 = 18$$

singers in all, so there are

$$\begin{aligned} {}^{18}C_6 &= \frac{18 \times 17 \times 16 \times 15 \times 14 \times 13}{6 \times 5 \times 4 \times 3 \times 2 \times 1} \\ &= 18\,564 \end{aligned}$$

ways of choosing six singers to form a sextet.

- (c) There are

$${}^5C_2 = 10 \text{ options for the pair of sopranos,}$$

$${}^6C_2 = 15 \text{ options for the pair of altos,}$$

$${}^3C_2 = 3 \text{ options for the pair of tenors,}$$

$${}^4C_2 = 6 \text{ options for the pair of basses.}$$

So, by the multiplication principle, there are

$$10 \times 15 \times 3 \times 6 = 2700$$

ways of choosing a pair from each voice group to form an octet.

- (d) For each of the ${}^4C_2 = 6$ pairs of voice groups, we use the multiplication principle to count the number of ways of choosing a singer from each voice group, as follows:

$$\begin{aligned} \text{soprano, alto: } &5 \times 6 = 30 \text{ options;} \\ \text{soprano, tenor: } &5 \times 3 = 15 \text{ options;} \\ \text{soprano, bass: } &5 \times 4 = 20 \text{ options;} \\ \text{alto, tenor: } &6 \times 3 = 18 \text{ options;} \\ \text{alto, bass: } &6 \times 4 = 24 \text{ options;} \\ \text{tenor, bass: } &3 \times 4 = 12 \text{ options.} \end{aligned}$$

Hence, by the addition principle, there are

$$30 + 15 + 20 + 18 + 24 + 12 = 119$$

ways of choosing a singer from each of two voice groups to sing a duet.

- (e) This part is similar to part (c), except that the order in which the two singers from each voice group are chosen is important. Hence there are

$${}^5P_2 = 20 \text{ options for the pair of sopranos,}$$

$${}^6P_2 = 30 \text{ options for the pair of altos,}$$

$${}^3P_2 = 6 \text{ options for the pair of tenors,}$$

$${}^4P_2 = 12 \text{ options for the pair of basses.}$$

So, by the multiplication principle, the number of ways to choose and arrange the eight singers in a line on the stage is

$$20 \times 30 \times 6 \times 12 = 43\,200.$$

(An alternative way to find the answer here is to use the fact that by the solution to part (c), there are 2700 ways to choose the singers. Then there are two ways to arrange the sopranos, two ways to arrange the altos, two ways to arrange the tenors, and two ways to arrange the basses. Hence, by the multiplication principle, the number of ways to choose and arrange the eight singers is

$$2700 \times 2^4 = 43\,200.)$$

- (f) There are $4!$ ways to order the four voice groups on the stage.

For each such order there are

$$5! \text{ ways to arrange the sopranos,}$$

$$6! \text{ ways to arrange the altos,}$$

$$3! \text{ ways to arrange the tenors,}$$

$$4! \text{ ways to arrange the basses.}$$

Hence, by the multiplication principle, there are

$$4! \times 5! \times 6! \times 3! \times 4! = 298\,598\,400$$

ways to arrange all the singers in their groups in a line on the stage.

Solution to Exercise 9

- (a) We can choose a 2-element subset of S in ${}^5C_2 = 10$ ways. For each such subset A , its complement $\overline{A}$ has three elements and is, of course, disjoint from A , with $S = A \cup \overline{A}$. Thus the set $S = \{1, 2, 3, 4, 5\}$ can be written as the union of a 2-element subset and a disjoint 3-element subset in the same number of ways as the number of 2-element subsets of S , namely ten.
- (b) A trial and error investigation of ways in which $\{1, 2, 3, 4\}$ can be written as the union of two disjoint 2-element subsets gives only three ways of doing this, namely

$$\begin{aligned} \{1, 2\} \cup \{3, 4\}, \\ \{1, 3\} \cup \{2, 4\}, \\ \{1, 4\} \cup \{2, 3\}. \end{aligned}$$

So the argument, which gave the number of ways as six, is incorrect – it's out by a factor of two. Where the argument goes wrong is that it double counts each way of expressing S as the union of two disjoint 2-element subsets. For instance, one of the 4C_2 2-element subsets of S is $\{1, 4\}$, and the argument associates this with expressing S as the union of disjoint 2-element subsets

$$\{1, 4\} \cup \overline{\{1, 4\}} = \{1, 4\} \cup \{2, 3\}.$$

However, another of the 4C_2 2-element subsets of S is $\{2, 3\}$, and the argument associates this with expressing S as the union of disjoint 2-element subsets

$$\{2, 3\} \cup \overline{\{2, 3\}} = \{2, 3\} \cup \{1, 4\},$$

which is the union of the same two subsets $\{1, 4\}$ and $\{2, 3\}$. So the correct answer can be obtained by halving the answer obtained from the argument.

This form of initial deliberate multiple counting followed by an appropriate adjustment is a useful technique.

- (c) Consider choosing three pairwise disjoint 2-element subsets of S , say A , B and C . There are 6C_2 ways to choose A . Once A

has been chosen, there are four elements of S left to choose from, so there are 4C_2 ways to choose B . Once A and B have been chosen, there is just one way to choose C . So, by the multiplication principle, the number of ways to choose the three sets is

$${}^6C_2 \times {}^4C_2 \times 1 = 15 \times 6 \times 1 = 90.$$

However, this count assumes that the order of the three sets is important. Since the order of the three sets is not important (for example, we want to consider $\{1, 2\} \cup \{3, 4\} \cup \{5, 6\}$ as the same as $\{3, 4\} \cup \{1, 2\} \cup \{5, 6\}$), we must divide the answer that we obtained from the count by the number of times that each way of expressing S has been counted.

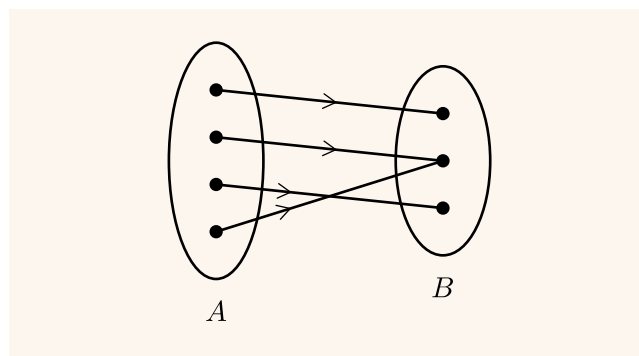
Each way $A \cup B \cup C$ of expressing S has been counted six times, namely as $A \cup B \cup C$, $A \cup C \cup B$, $B \cup A \cup C$, $B \cup C \cup A$, $C \cup A \cup B$ and $C \cup B \cup A$. This is because the number of times that each way has been counted is the number of ways of ordering three sets A , B and C , and this is the number of permutations of three objects taken three at a time, which is $3! = 6$.

So the number of ways in which $\{1, 2, 3, 4, 5, 6\}$ can be written as the union of three pairwise disjoint 2-element subsets is

$$\frac{90}{6} = 15.$$

Solution to Exercise 10

For such a function, we need a pair of elements of A to map to the same element of B , and the remaining two elements of A to map one-to-one to the remaining two elements of B , as shown in the diagram below.



Exercise Booklet 12

We can select the pair of elements in A in 4C_2 ways, and there are three options for its image in B . There are two ways of mapping the remaining two elements of A one-to-one to the remaining two elements of B . So by the multiplication principle, there are

$${}^4C_2 \times 3 \times 2 = 6 \times 3 \times 2 = 36$$

functions from A to B with image set B .

Solution to Exercise 11

If the team consists of two French specialists, one German specialist and one Spanish specialist, then the number of ways to choose it is

$${}^3C_2 \times {}^2C_1 \times {}^2C_1 = 3 \times 2 \times 2 = 12.$$

If it consists of one French specialist, two German specialists and one Spanish specialist, then the number of ways to choose it is

$${}^3C_1 \times {}^2C_2 \times {}^2C_1 = 3 \times 1 \times 2 = 6.$$

If it consists of one French specialist, one German specialist and two Spanish specialists, then the number of ways to choose it is

$${}^3C_1 \times {}^2C_1 \times {}^2C_2 = 3 \times 2 \times 1 = 6.$$

Hence, by the addition principle, the total number of ways to choose the team is

$$12 + 6 + 6 = 24.$$

(Alternatively, you could consider choosing the team by first choosing one specialist from each language, and then choosing a fourth teacher to complete the team. However, with this method, you count each possible team twice (because, for example, if a particular team contains two French specialists, say A and B, then this team could have been chosen either by choosing A as the original French teacher and B as the fourth teacher, or the other way round). So, to obtain the correct answer, you must halve the number that you obtain from your count. The number of ways to choose a specialist from each language is $3 \times 2 \times 2 = 12$, and the number of ways to choose a fourth teacher from the four teachers not yet chosen is 4. Hence the number of ways to choose the team is $\frac{1}{2}(12 \times 4) = 24$.)

Solution to Exercise 12

Three clubs can be selected by first choosing three different days of the week, then choosing a club available on the first day, then a club available on the second day, and finally a club available on the third day.

The order in which the days of the week are chosen does not matter, so the number of ways to choose the three days is ${}^5C_3 = 10$.

For each particular day of the week, there are 4 ways to choose a club.

Hence, by the multiplication principle, the total number of different ways for a pupil to choose three clubs is

$$10 \times 4^3 = 640.$$

Solution to Exercise 13

In each question part, the sample space S can be taken to consist of triples of numbers, where the first number is the score on the first die, the second number is the score on the second die and the third number is the score on the third die. That is,

$$\begin{aligned} S = \{ & (1, 1, 1), (1, 1, 2), \dots, (1, 1, 6), \\ & (1, 2, 1), (1, 2, 2), \dots, (1, 2, 6), \\ & \vdots \\ & (2, 1, 1), (2, 1, 2), \dots, (2, 1, 6), \\ & (2, 2, 1), (2, 2, 2), \dots, (2, 2, 6), \\ & \vdots \\ & (6, 5, 1), (6, 5, 2), \dots, (6, 5, 6), \\ & (6, 6, 1), (6, 6, 2), \dots, (6, 6, 6) \}. \end{aligned}$$

By the multiplication principle, the size of the sample space S is

$$|S| = 6 \times 6 \times 6 = 216.$$

(a) Getting exactly three 5s is the event

$$E = \{(5, 5, 5)\},$$

with $|E| = 1$. Thus the probability of getting exactly three 5s is

$$P(E) = \frac{1}{216}.$$

- (b) We can calculate the size of the event E of getting at least one 5 by using the subtraction principle. Consider the event $\overline{E}$ that is the complement of E . This event consists of all triples in S containing no 5, which corresponds to each die showing one of the five numbers in $\{1, 2, 3, 4, 6\}$. By the multiplication principle,

$$|\overline{E}| = 5 \times 5 \times 5 = 125.$$

Hence, by the subtraction principle,

$$|E| = |S| - |\overline{E}| = 216 - 125 = 91.$$

Therefore the probability that there is at least one 5 is

$$P(E) = \frac{91}{216}.$$

- (c) The event that the total score is 5 is

$$E = \{(1, 1, 3), (1, 2, 2), (1, 3, 1), (2, 1, 2), (2, 2, 1), (3, 1, 1)\}.$$

Since $|E| = 6$, the probability of getting a total score of 5 is

$$P(E) = \frac{6}{216} = \frac{1}{36}.$$

- (d) We can calculate the size of the event E that the total score is at least 5 by using the subtraction principle. Consider the event $\overline{E}$ that is the complement of E . This event consists of all triples in S with total score less than 5, namely

$$\overline{E} = \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (2, 1, 1)\}.$$

Since $|\overline{E}| = 4$, by the subtraction principle we have

$$|E| = |S| - |\overline{E}| = 216 - 4 = 212.$$

Therefore the probability that the total score is at least 5 is

$$P(E) = \frac{212}{216} = \frac{53}{54}.$$

Solution to Exercise 14

In each question part, the sample space S can be taken to comprise all six-letter sequences made up of the letters H and T .

The size of this sample space S is $2^6 = 64$.

- (a) The event is $E = \{TTTTTT\}$. Therefore the probability that the coin lands showing tails every time is

$$P(E) = \frac{1}{64}.$$

- (b) The size of the event E that at least one toss of the coin shows tails can be calculated using the subtraction principle. Consider the event $\overline{E}$ that is the complement of E . This consists of all outcomes in S such that *no* toss of the coin shows tails, which is the same as saying that all six tosses show heads.

So $\overline{E} = \{HHHHHH\}$. By the subtraction principle,

$$|E| = |S| - |\overline{E}| = 64 - 1 = 63.$$

Therefore the probability that at least one toss of the coin shows tails is

$$P(E) = \frac{63}{64}.$$

- (c) The event E consists of the outcomes in which a T appears in just one of the six positions, with H s everywhere else. So $|E| = 6$, and therefore the probability that exactly one toss of the coin shows tails is

$$P(E) = \frac{6}{64} = \frac{3}{32}.$$

- (d) In part (b), it was found that there are 63 outcomes with at least one tail. In part (c), it was found that there are 6 outcomes with exactly one tail. It follows that the number of outcomes with at least two tails is $63 - 6 = 57$. Therefore the probability that at least two tosses of the coin show tails is

$$P(E) = \frac{57}{64}.$$

Solution to Exercise 15

- (a) A closed form is

$$x_n = C \times \left(\frac{5}{2}\right)^{n-1} + D,$$

for some constants C and D . The first term is $x_1 = -4$, and the second term is

$$x_2 = \frac{5}{2} \times (-4) + 8 = -2.$$

So

$$\begin{aligned} -4 &= C + D, \\ -2 &= \frac{5}{2}C + D. \end{aligned}$$

Subtracting the first equation from the second gives $2 = \frac{3}{2}C$, so $C = \frac{4}{3}$.

Substituting this into the first equation gives $D = -4 - \frac{4}{3} = -\frac{16}{3}$. Therefore a closed form for the system is

$$\begin{aligned} x_n &= \frac{4}{3} \left(\frac{5}{2}\right)^{n-1} - \frac{16}{3} \\ (n &= 1, 2, 3, \dots). \end{aligned}$$

Exercise Booklet 12

- (b) A closed form is

$$u_n = C \times \left(-\frac{1}{2}\right)^{n-1} + D,$$

for some constants C and D . The first term is $u_1 = 6$, and the second term is

$$u_2 = -\frac{1}{2} \times 6 + 3 = 0.$$

So

$$6 = C + D,$$

$$0 = -\frac{1}{2}C + D.$$

Subtracting the second equation from the first gives $6 = \frac{3}{2}C$, so $C = 4$. Substituting this into the first equation gives $D = 6 - 4 = 2$. Therefore a closed form for the system is

$$u_n = 4\left(-\frac{1}{2}\right)^{n-1} + 2 \quad (n = 1, 2, 3, \dots).$$

(Since

$$4 = 2^2 = \left(\left(\frac{1}{2}\right)^{-1}\right)^2 = \left(\frac{1}{2}\right)^{-2} = \left(-\frac{1}{2}\right)^{-2},$$

which gives

$$4\left(-\frac{1}{2}\right)^{n-1} = \left(-\frac{1}{2}\right)^{-2}\left(-\frac{1}{2}\right)^{n-1} = \left(-\frac{1}{2}\right)^{n-3},$$

the closed form above can also be written as

$$u_n = \left(-\frac{1}{2}\right)^{n-3} + 2 \quad (n = 1, 2, 3, \dots).$$

Solution to Exercise 16

- (a) For $n \geq 2$, the term x_n is calculated by doubling the term x_{n-1} and then subtracting 1. So a recurrence system is

$$x_1 = 1.25, \quad x_n = 2x_{n-1} - 1 \quad (n = 2, 3, 4, \dots).$$

- (b) As the recurrence system is first-order, a closed form for the infinite sequence (x_n) that it gives is

$$x_n = C \times 2^{n-1} + D,$$

for some constants C and D . The first term is $x_1 = 1.25$, and the second term is

$$x_2 = 1.25 \times 2 - 1 = 1.5.$$

So

$$1.25 = C + D,$$

$$1.5 = 2C + D.$$

Subtracting the first equation from the second gives $0.25 = C$, and substituting this into the first equation gives

$D = 1.25 - 0.25 = 1$. Therefore a closed form for the system is

$$x_n = 0.25 \times 2^{n-1} + 1 \quad (n = 1, 2, 3, \dots).$$

Since

$$\begin{aligned} 0.25 \times 2^{n-1} &= \frac{1}{4} \times 2^{n-1} \\ &= 2^{-2} \times 2^{n-1} \\ &= 2^{n-3}, \end{aligned}$$

the closed form can be written more simply as

$$x_n = 2^{n-3} + 1 \quad (n = 1, 2, 3, \dots).$$

- (c) From the closed form found in part (b), we can see that x_n increases as n increases. To find the day when the investment will first exceed £1000, we start by solving the equation $2^{n-3} + 1 = 1000$ for a non-integer value of n . This equation can be written as

$$2^{n-3} = 999,$$

which gives

$$n - 3 = \log_2 999,$$

and hence

$$n = \log_2 999 + 3 = 12.96 \dots$$

Since $2^{n-3} + 1$ increases as n increases, it follows that $2^{12-3} + 1 < 1000$ and $2^{13-3} + 1 > 1000$; that is, $x_{12} < 1000$ and $x_{13} > 1000$. So the investment will exceed £1000, and it first does this on the 13th day.

(Check:

$$x_{12} = 2^{12-3} + 1 = 2^9 + 1 = 513$$

and

$$x_{13} = 2^{13-3} + 1 = 2^{10} + 1 = 1025.)$$

- (d) If the initial investment is changed to £1, then the recurrence system becomes

$$x_1 = 1, \quad x_n = 2x_{n-1} - 1 \quad (n = 2, 3, 4, \dots).$$

Calculating the second term gives

$$x_2 = 2x_1 - 1 = 2 \times 1 - 1 = 1.$$

So $x_n = 1$ for all $n = 1, 2, 3, \dots$. That is, the investment stays at £1 forever.

Solution to Exercise 17

(a) The auxiliary equation is

$$r^2 + \frac{11}{3}r - \frac{4}{3} = 0,$$

that is,

$$3r^2 + 11r - 4 = 0.$$

Factorising gives

$$(3r - 1)(r + 4) = 0.$$

This has solutions $r = \frac{1}{3}$ and $r = -4$.

Thus the general solution of the recurrence relation is

$$u_n = A \times \left(\frac{1}{3}\right)^n + B \times (-4)^n,$$

where A and B are unknown constants.

To find A and B , we use the initial terms of the sequence, $u_0 = 9$ and $u_1 = -10$.

Substituting $n = 0$ and $u_0 = 9$ into the general solution, we obtain

$$9 = A \times \left(\frac{1}{3}\right)^0 + B \times (-4)^0,$$

so

$$A + B = 9.$$

Similarly, substituting $n = 1$ and $u_1 = -10$ gives

$$-10 = A \times \left(\frac{1}{3}\right)^1 + B \times (-4)^1,$$

so

$$\frac{1}{3}A - 4B = -10.$$

Subtracting three times the second equation from the first equation gives $13B = 39$, so $B = 3$. Then substituting this value of B into the first equation gives $A = 6$.

Thus a closed form for the given recurrence sequence is

$$u_n = 6\left(\frac{1}{3}\right)^n + 3(-4)^n \quad (n = 0, 1, 2, \dots).$$

(b) The auxiliary equation is

$$r^2 + \frac{2}{3}r + \frac{1}{9} = 0,$$

that is,

$$9r^2 + 6r + 1 = 0.$$

Factorising gives

$$(3r + 1)^2 = 0.$$

This has the (repeated) solution $r = -\frac{1}{3}$.

Thus the general solution of the recurrence relation is

$$u_n = (A + Bn) \times \left(-\frac{1}{3}\right)^n,$$

where A and B are unknown constants.

To find A and B , we use the initial terms of the sequence, $u_0 = 5$ and $u_1 = -1$.

Substituting $n = 0$ and $u_0 = 5$ into the general solution, we obtain

$$5 = (A + B \times 0) \times \left(-\frac{1}{3}\right)^0,$$

so $A = 5$.

Similarly, substituting $n = 1$ and $u_1 = -1$ gives

$$-1 = (A + B \times 1) \times \left(-\frac{1}{3}\right)^1,$$

so

$$3 = A + B.$$

Substituting $A = 5$ into this equation then gives $B = -2$.

Thus a closed form for the given recurrence sequence is

$$u_n = (5 - 2n)\left(-\frac{1}{3}\right)^n \quad (n = 0, 1, 2, \dots).$$

(c) The auxiliary equation is

$$r^2 - \frac{7}{2}r - 2 = 0,$$

that is,

$$2r^2 - 7r - 4 = 0.$$

Factorising gives

$$(2r + 1)(r - 4) = 0.$$

This has solutions $r = -\frac{1}{2}$ and $r = 4$.

Thus the general solution of the recurrence relation is

$$x_n = A \times \left(-\frac{1}{2}\right)^n + B \times 4^n,$$

where A and B are unknown constants.

To find A and B , we use the initial terms of the sequence, $x_0 = -1$ and $x_1 = 14$.

Substituting $n = 0$ and $x_0 = -1$ into the general solution, we obtain

$$-1 = A \times \left(-\frac{1}{2}\right)^0 + B \times 4^0,$$

so

$$A + B = -1.$$

Exercise Booklet 12

Similarly, substituting $n = 1$ and $x_1 = 14$ gives

$$14 = A \times \left(-\frac{1}{2}\right)^1 + B \times 4^1,$$

so

$$-\frac{1}{2}A + 4B = 14.$$

Adding twice the second equation to the first equation gives $9B = 27$, so $B = 3$. Substituting this value of B into the first equation then gives $A = -4$.

Thus a closed form for the given recurrence sequence is

$$x_n = -4\left(-\frac{1}{2}\right)^n + 3 \times 4^n \\ (n = 0, 1, 2, \dots).$$

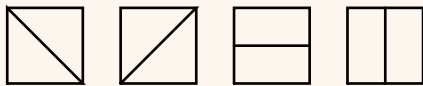
Solution to Exercise 18

- (a) For $n = 1$, the only possible path is a single rectangular slab, as shown below.



So $u_1 = 1$.

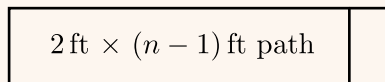
For $n = 2$, there are four possibilities.



So $u_2 = 4$.

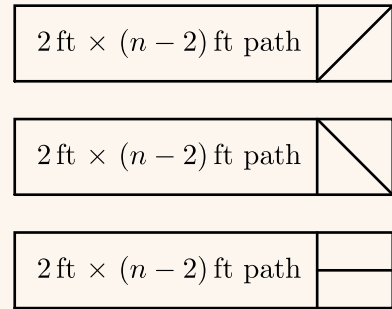
- (b) Consider any $2 \text{ ft} \times n \text{ ft}$ path with $n \geq 3$. There are four possibilities for the last (rightmost) slab(s).

The path could end with a single 'vertical' rectangular slab, as shown below.



There are u_{n-1} such paths.

Alternatively, the path could end with a square formed by a pair of triangular slabs or by a pair of 'horizontal' rectangular slabs, as shown below.



There are u_{n-2} paths of each of these three types.

Thus, by the addition principle, the number of $2 \text{ ft} \times n \text{ ft}$ paths with $n \geq 3$ is given by

$$u_n = u_{n-1} + u_{n-2} + u_{n-2} + u_{n-2} \\ = u_{n-1} + 3u_{n-2}.$$

- (c) The sequence (u_n) satisfies the recurrence system

$$u_1 = 1, \quad u_2 = 4, \quad u_n = u_{n-1} + 3u_{n-2} \\ (n = 3, 4, 5, \dots).$$

The auxiliary equation for the recurrence relation is

$$r^2 - r - 3 = 0,$$

which has solutions

$$r = \frac{1 \pm \sqrt{1 - 4 \times 1 \times (-3)}}{2} \\ = \frac{1 \pm \sqrt{13}}{2}.$$

Therefore the general solution of the recurrence relation is

$$u_n = A \left(\frac{1 + \sqrt{13}}{2} \right)^n + B \left(\frac{1 - \sqrt{13}}{2} \right)^n,$$

where A and B are constants.

The first and second terms of the sequence are presently $u_1 = 1$ and $u_2 = 4$. To find the 0th term, rearrange the recurrence relation in the form

$$u_{n-2} = \frac{1}{3}(u_n - u_{n-1}).$$

Since $u_1 = 1$ and $u_2 = 4$, we obtain

$$u_0 = \frac{1}{3}(u_2 - u_1) = \frac{1}{3}(4 - 1) = 1.$$

To find A and B , we now use the initial terms of the sequence, $u_0 = 1$ and $u_1 = 1$.

Substituting $n = 0$ and $u_0 = 1$ into the general solution, we obtain

$$1 = A \left(\frac{1 + \sqrt{13}}{2} \right)^0 + B \left(\frac{1 - \sqrt{13}}{2} \right)^0,$$

that is,

$$A + B = 1.$$

Similarly, substituting $n = 1$ and $u_1 = 1$ gives

$$1 = A \left(\frac{1 + \sqrt{13}}{2} \right)^1 + B \left(\frac{1 - \sqrt{13}}{2} \right)^1,$$

that is,

$$A \left(\frac{1 + \sqrt{13}}{2} \right) + B \left(\frac{1 - \sqrt{13}}{2} \right) = 1.$$

Subtracting the second equation from $(1 + \sqrt{13})/2$ times the first equation gives

$$B \left(\frac{1 + \sqrt{13}}{2} - \frac{1 - \sqrt{13}}{2} \right) = \frac{1 + \sqrt{13}}{2} - 1,$$

which gives

$$B\sqrt{13} = \frac{\sqrt{13} - 1}{2}.$$

Hence

$$B = \frac{\sqrt{13} - 1}{2\sqrt{13}} = \frac{1}{26}(13 - \sqrt{13}).$$

Substituting this value of B into the first equation gives

$$\begin{aligned} A &= 1 - B \\ &= 1 - \frac{1}{26}(13 - \sqrt{13}) \\ &= \frac{1}{26}(13 + \sqrt{13}). \end{aligned}$$

Thus the number of $2 \text{ ft} \times n \text{ ft}$ paths is given by

$$\begin{aligned} u_n &= \frac{1}{26}(13 + \sqrt{13}) \left(\frac{1 + \sqrt{13}}{2} \right)^n \\ &\quad + \frac{1}{26}(13 - \sqrt{13}) \left(\frac{1 - \sqrt{13}}{2} \right)^n \\ &\quad (n = 1, 2, 3, \dots). \end{aligned}$$

(This formula can be written a little more concisely as

$$\begin{aligned} u_n &= \frac{1}{\sqrt{13}} \left(\left(\frac{1 + \sqrt{13}}{2} \right)^{n+1} - \left(\frac{1 - \sqrt{13}}{2} \right)^{n+1} \right) \\ &\quad (n = 1, 2, 3, \dots). \end{aligned}$$